Hyperkaleidocycles

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Wanted: a toy-maker who is comfortable constructing games in the fourth-dimension. A new toy has been discovered, and is waiting to find expression!

On your last visit to the toy-store, you probably saw a *kaleidocycle*. A kaleidocycle is a closed ring of polyhedra where every two neighboring polyhedra share one common edge which acts as a hinge. The most common type is the *closed tetrahedral kaleidocycle*, pictured below in Figure 1 as it goes through its rotation.

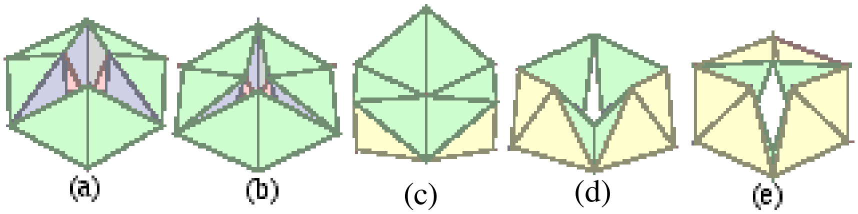

(a) (b) (c) (d) (e)

Figure 1

Many different kinds of kaleidocycles exist. Creating them out of cardboard or wood is a wonderful past-time. In this article, I will posit the existence of, and discuss the rotation of a class of four-dimensional analogues to the kaleidocycle, which I call the *hyperkaleidocycle*.

Introduction

Three-dimensional kaleidocycles exist in a variety of forms. For example, eight cubes can be hinged in such a way that in every configuration, the ring has one degree of rotational freedom. Such a ring is called a ring of *Yoshi* Cubes.

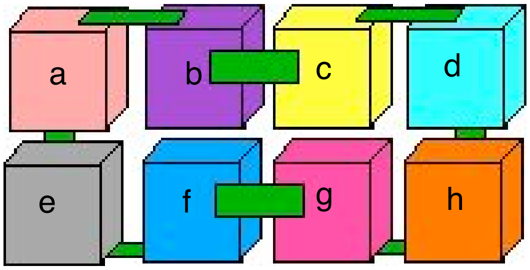
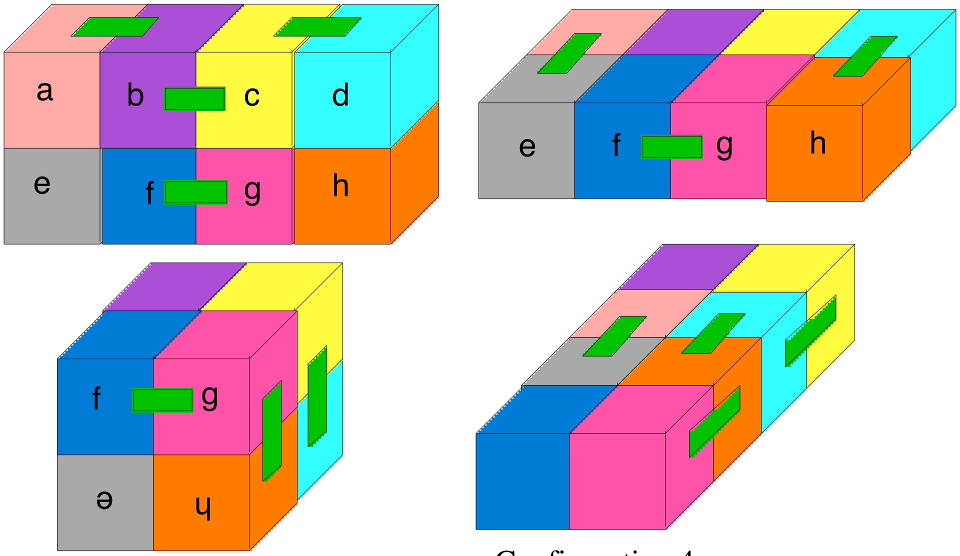


Figure 2
A set of Yoshi Cubes with all hinges visible.

Figure 1 shows the set of eight cubes with one face of each cube lettered a through h. In the diagram, each cube is separated a bit from its neighbors so that the hinging pattern can be seen. The green parallelograms represent the hinged connections.

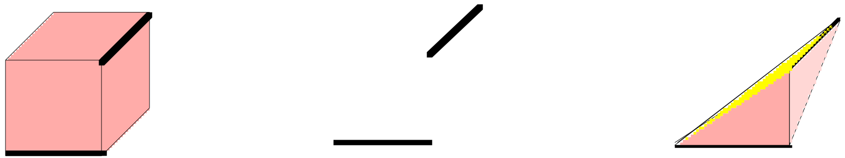
Figure 3 shows four different configurations of the same set of Yoshi cubes -- that is, three of the rotations that can be performed on the Yoshi Cubes. After three additional rotations, the cubes are back in their original configuration.



Configuration 1 Configuration 2 Configuration 3 Configuration 4

Figure 3
Rotating a set of Yoshi Cubes

A quick examination of the ring of Yoshi cubes reveals that only two edges on each cube are attached to neighboring cubes, and that the edges used as hinges on each cube are orthogonal to each other. In figure 4, one of the Yoshi cubes has been separated from the ring. Figure 3a shows the cube with the two orthogonal "hinge" edges in bold. Figure 4b shows the two orthogonal edges, with the cube subtracted. Finally in figure 4c, a tetrahedron has been formed from the two hinge edges of the original Yoshi cube. This tetrahedron contains two faces which are isosceles-right triangles.



(a) (b) (c)

Figure 4
"Yoshi" Tetrahedron

Building eight such tetrahedra and attaching them together edge-to-edge in the same manner as were the cubes from which they were derived creates a kaleidocycle that rotates in the same manner as the original ring of Yoshi cubes.

Kaleidocycles made of rings of tetrahedra may be categorized into various classes. For example, the ring of "Yoshi" tetrahedra described above falls under the class of *normal kaleidocycles*. In contrast, r*egular kaleidocycles* are made of rings of regular tetrahedra. For an elegant discussion of the behavior of the various classes of kaleidocycles, see http://www.kaleidocycles.de/theory.shtml, posted by M. Engel on May 7, 2003. My approach to classifying hyperkaleidocycles is based on Mr. Engels's work in three dimensions.

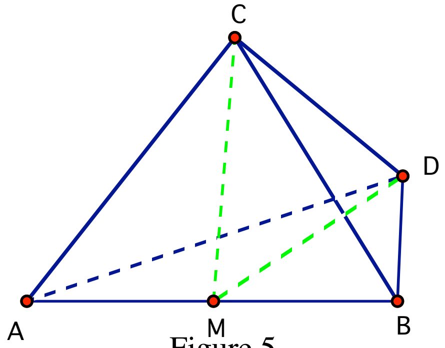


Figure 5

Of the different classes of kaleidocycles, the one most commonly found in toy stores is the *closed kaleidocycle*. A closed kaleidocycle consists of six irregular tetrahedra arranged in a ring, as seen in Figure 1. The tetrahedra must be created from isosceles triangles whose altitudes from their vertex angles are equal to the length of their bases. Such a tetrahedron is pictured in Figure 5.

In Figure 5, edges [AB] and [CD] are orthogonal to each other, and each of unary length. M represents the midpoint of edge [AB]. [CM] is the altitude of face ∏ ABC and [DM] is the altitude of face ∏ ABD. The two altitudes are shown as green dashed lines.

To create a closed kaleidocycle, it is necessary that CM = DM = AB = CD. Similarly, the altitudes of triangles ∏ CDB and ∏ CDA must be equal in length to edges [AB] and [CD].

The two orthogonal edges [AB] and [CD] become the hinge-edges to which neighboring tetrahedra are attached. Creating a ring of six such tetrahedra creates a kaleidocycle that has the interesting property that for various rotations, the central “hole” of the ring closes up.

Figure 1 shows a closed kaleidocycle rotating through its closed position. Notice that in its closed position (Figure 1c) the kaleidocycle projects onto a regular hexagon. That is why the altitudes of the faces must be equal in length to the hinge edges.

In this paper, I will discuss a variety of classes of hyperkaleidocycles -- rings of simplices, or four-dimensional polytopes. My ultimate goal is to describe a hyperkaleidocycle with the distinguishing property of a closed kaleidocycle: that there is a rotation in which the central hole of the ring disappears.

Toward that end, consider a ring consisting of regular simplices. A simplex, also called a 5-cell, is the simplest of the regular polytopes. It is a four dimensional, convex object with five vertices, ten edges, ten faces, and five bounding tetrahedra. In a regular simplex, all edges are congruent, as are all faces and all tetrahedra.

I. The Basic Unit of the Regular Hyperkaleidocycle

Let A, B, C, D, and E be the vertices of a regular simplex. Let M be the midpoint of edge [AB]. Let P be the centroid of face ∏ ABC, and Q be the midpoint of edge [DE]. Below are two views of such a simplex. Figure 6 represents an ^{3} diagram showing part of simplex ABCDE -- namely, tetrahedron ABCD, with points M and P included. Visible edges are in bold blue, obscured edges are dashed, and the connection between midpoint M and centroid P is in green.

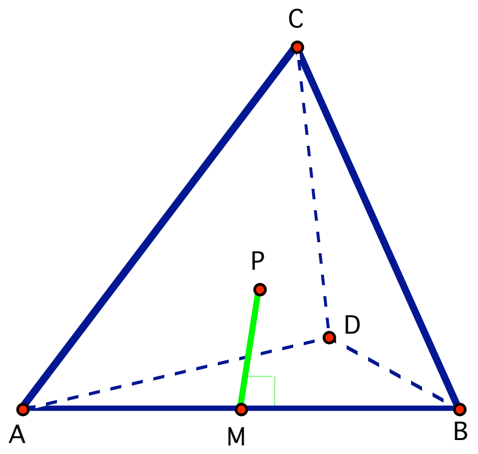


Figure 6

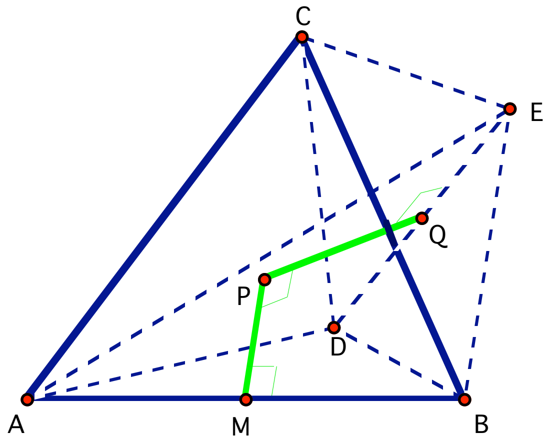


Figure 7

Figure 7 shows the full ^{4} simplex ABCDE in blue with the green segments from midpoint M to centroid P, and from centroid P to midpoint Q included.

I will now show that

$$[AB]⊓ [CM]⊓ [PQ]⊓ [DE] \qquad (1)$$

Clearly, [AB]⊓ [CM] as ⊓ ABC is an equilateral triangle. P, the centroid of ⊓ ABC, lies on [CM]. Tetrahedron ABCD is regular with height [DP]. Similarly, tetrahedron ABCE is regular with height [EP]. Since the tetrahedra are congruent, DP = EP. Then, as Q is the midpoint of [DE], the points Q and P are both equidistant from the endpoints of segment [DE], and thus determine the perpendicular bisector of it. Therefore, [PQ]⊓ [DE], and, in particular, ⊓ PDQ is a right triangle with right angle at ⊓ Q.

It remains to be proven that [CM]⊓ [PQ]. Consider the following lengths, where *s* represents the length of an edge of the simplex:

CM = $s\sqrt{3}/2$, as it is the altitude of equilateral ⊓ ABC. Similarly,
CQ = $s\sqrt{3}/2$ as it is the altitude of equilateral ⊓ CDE.
MP = $s\sqrt{3}/6$, as the centroid of an equilateral triangle is one-third of the way from its base to opposite vertex.
PD = $s\sqrt{6}/3$, as it is the altitude of regular tetrahedron ABCD.
DQ = $s/2$, as Q is the midpoint of [DE].
PQ = $s\sqrt{15}/6$, since, as ⊓ PDQ is right, $(PD)^2 = (DQ)^2 + (PQ)^2$.
CP = $s\sqrt{3}/3$, which is CM - MP.

Having calculated these lengths, it is easy to see that $(CP)^2 + (PQ)^2 = (CQ)^2$, so ⊓ CPQ is a right triangle with right angle at ⊓ P, meaning that [CM] ⊓ [PQ].

For ease, let PQ = *h*, so that when the simplex is regular,

$$h = s\sqrt{15}/6 \text{, or } s = 2h\sqrt{15}/5. \qquad (2)$$

Finally, it holds that in a regular simplex, an edge between any two vertices is orthogonal to each of the three edges connecting the remaining three vertices. This is because every grouping of four vertices defines a regular tetrahedron, and in a regular tetrahedron, opposite edges are orthogonal.

II. Definition of the Bounding Spaces of Rotation

Let $n \square\square$, $n \geq 8$, and $\square = 2\square/n$. Then $0 < \square \leq \square/4$. Furthermore, let

$S_0 = \{(x, y, z, w) \mid\mid {}^4 \mid w = 0\}$ represent the x-y-z coordinate 3-space,

and let

$S_{\square} = \{(x, y, z, w) \mid\mid {}^4 \mid w = x \tan \square\}$

$\overrightarrow{n_{\square}} = (-\sin\square, 0, 0, \cos\square)$.

The 3-spaces S_0 and $S_{\square}$ intersect in the y-z plane. The angle between them is $\square$. The vector $\overrightarrow{n_{\square}}$ is a normal vector to the 3-space $S_{\square}$. I will later show that a simplex can rotate in such a manner that it stays within the bounding spaces S_0 and $S_{\square}$.

III. Position the First Simplex within the Bounding Spaces

Position a regular simplex $\square$ (notations as in **I**) as follows:

(i) Q lies in the x-w plane (that is, $y = z = 0$).
(ii) P lies on the positive x-axis (that is $y = z = w = 0$).
(iii) A, B, C, P, and M $\mid\mid$ S_0 (that is, $w = 0$).
(iv) D, Q, and E $\mid\mid$ $S_{\square}$ (that is, $w = x \tan \square$).
(v) D, Q, and E have positive x-coordinate.

Figure 8 is a representation of a simplex laid out satisfying conditions (i) through (v). The box on the left represents the 3-space S_0 and the box on the right represents the 3-space $S_{\square}$, as defined in **II** above. The origins of the two boxes should be considered to be coincident. A, B, C, D, and E are the vertices of the simplex (as defined in **I** above. Note that for simplicity, edges [DB], [DA], and [EA] are not shown. The remaining edges of the simplex are shown in bold blue.) P is the centroid of face $\square$ ABC, and Q is the midpoint of edge [DE]. Finally, as it plays an important role in the upcoming discussion, [PQ], which is *not* an edge of the simplex, is shown as a dotted red line.

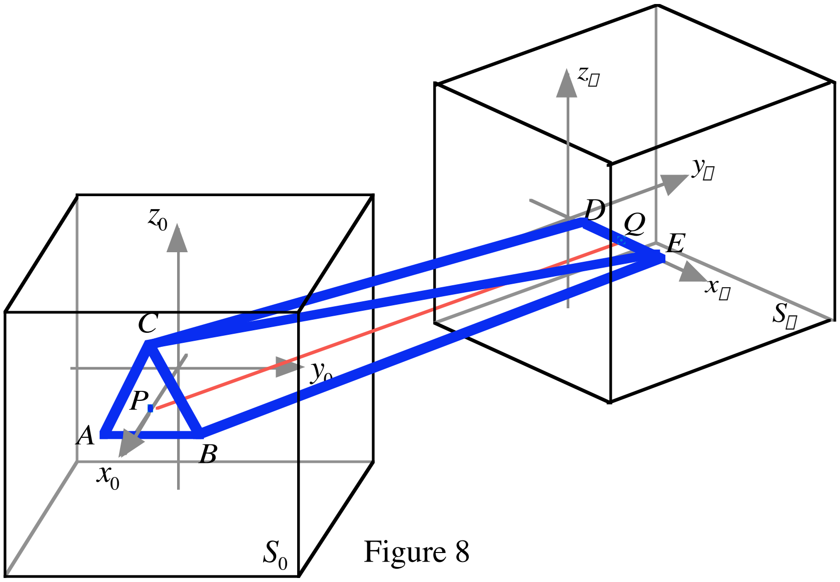


Figure 8

From $0 < \square \leq \square/4$ and (1), such a simplex exists and is uniquely determined by s and $\square$. To illustrate a simplex so positioned, consider

$\overrightarrow{AB} = (0, s, 0, 0)$ satisfies (iii)
$\overrightarrow{AC} = \left(0, s/2, s\sqrt{3}/2, 0\right)$ satisfies (iii)
$\overrightarrow{BC} = \left(0, \square s/2, s\sqrt{3}/2, 0\right)$ satisfies (iii).
Together, $\overrightarrow{AB}$, $\overrightarrow{AC}$, and $\overrightarrow{BC}$ satisfy (ii).

Now consider $\overrightarrow{DE} = (p, q, r, p\tan\square)$, which satisfies (iv) for any $0 < \square \leq \square/4$. $\overrightarrow{DE}$ is orthogonal to $\overrightarrow{AB}$, $\overrightarrow{AC}$, and $\overrightarrow{BC}$ (from **I** above.) Therefore

q = 0 as $\overrightarrow{DE} \bullet \overrightarrow{AB} = 0$, and
r = 0 as $\overrightarrow{DE} \bullet \overrightarrow{AC} = 0$.

Recalling that the length of each side of the regular simplex is s, $\|\overrightarrow{DE}\| = s$,

$p^2 + q^2 + r^2 + (p\tan\square)^2 = s^2$, which simplifies to $p^2 + p^2\tan^2\square = s^2$, or $p^2\sec^2\square = s^2$.

When taking the square root, the positive value of p must be chosen (since $s > 0$, and $0 < \square \leq \square/4$) to get $p = s\cos\square$. Combining these results, yields:

$\overrightarrow{AB} = (0, s, 0, 0)$
$\overrightarrow{AC} = \left(0, s/2, s\sqrt{3}/2, 0\right)$
$\overrightarrow{BC} = \left(0, \square s/2, s\sqrt{3}/2, 0\right)$
$\overrightarrow{DE} = \left(s\cos\square, 0, 0, s\sin\square\right)$ satisfies (i), (iv) and (v).

A simplex laid out satisfying conditions (i) through (v) has the added property that A, B, and C have positive x-coordinates, as $OP > OQ$ (where O represents the origin.)

To see why, consider Figure 9 which shows the configuration of D, Q, E, and P with respect to the origin. Note that the two axes shown are the x- and w-axes. D, Q, and E all lie along the line $w = x\tan\theta$ (condition (iv).) From (1), [DE] ⊥ [PQ]. Therefore OP = h/sinθ. Since $0 < \theta \leq \pi/4$, $0 < \sin\theta \leq \sqrt{2}/2$, so h/sin$\theta \geq h\sqrt{2}$. Combining this result with (2) yields $OP = h/\sin\theta \geq h\sqrt{2} = s\sqrt{30}/6 > s\sqrt{3}/3$ = AP = BP = CP, which demonstrates that not only are the x-coordinates of points A, B, and C positive, but that there is enough clearance for the simplex to rotate about P without the x-coordinates of A, B, or C becoming negative.

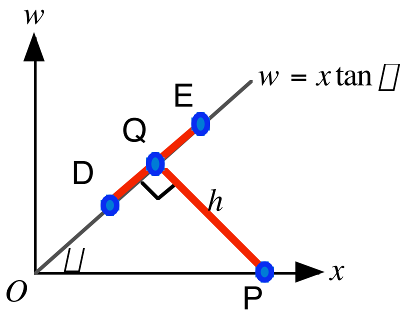


Figure 9

Also note that

the vectors $\overrightarrow{AB}, \overrightarrow{MC}, \vec{n}, \overrightarrow{DE}$ form a right-handed system. (3)

the vectors $\overrightarrow{AB}, \overrightarrow{MC}, \overrightarrow{PQ}, \overrightarrow{DE}$ form a right-handed system. (4)

In **V**, I will show that a simplex positioned as above can be rotated in any direction in S_0 without violating conditions (ii) through (v) above.

IV. Reflect the Simplex to Create a Ring

Further restrict n so that n ≥ 8, and n is even. Reflecting the simplex □ about the space S_θ yields another simplex $□_2$, that shares vertices D and E with □. This can be done for each vertex of the simplex by rotating the point about the y-z plane by an angle of $-\theta$, negating the resulting w-coordinate of the point, and rotating it back about the y-z plane by an angle of θ. By successively rotating □ and $□_2$ about the y-z plane by an angle of 2θ, further simplices are obtained (altogether n simplices) that form (because n is even and $\theta = 2\pi/n$) a closed ring where every two neighboring simplices share either one common edge (in the case of □ and $□_2$, edge [DE]) or one common face (in the case of □ and $□_n$, face △ABC.) I call this ring a *regular hyperkaleidocycle*.

V. Rotation Occurs within the Bounding Spaces

I now demonstrate how □ can be rotated within the delimiting spaces S_0 and S_θ while conditions (ii) through (v) from **III** remain fulfilled. Then by symmetry, it follows that a ring of simplices assembled as in **IV** can be inverted while the property that neighboring simplices share either a common face or a common edge is preserved.

Let the parameter $\theta \in [0, 2\pi)$ describe the rotation of the simplex $\square$ about the y-w plane in the sense that θ specifies the actual angle between $\overrightarrow{CM}$ and the positive z-axis. Let the parameter $t \in [0, 2\pi)$ describe the rotation of the simplex $\square$ about the z-w plane in the sense that t specifies the actual angle between $\overrightarrow{AB}$ and the positive y-axis.

Let $A_{\theta,t}$, $B_{\theta,t}$, etc. denote the positions of the corresponding points for rotations of θ and t (in that order).

Consider

$$\vec{u} = \frac{\overrightarrow{A_{\theta,t}B_{\theta,t}}}{\left\|\overrightarrow{A_{\theta,t}B_{\theta,t}}\right\|} = \frac{1}{\left\|\overrightarrow{A_{\theta,t}B_{\theta,t}}\right\|}\cdot(0,s,0,0)\begin{pmatrix}\cos\theta & 0 & \sin\theta & 0\\ 0 & 1 & 0 & 0\\ -\sin\theta & 0 & \cos\theta & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\cos t & \sin t & 0 & 0\\ -\sin t & \cos t & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}$$

$$= (-\sin t, \cos t, 0, 0) \in S_0$$

and

$$\vec{g} = \frac{\overrightarrow{M_{\theta,t}C_{\theta,t}}}{\left\|\overrightarrow{M_{\theta,t}C_{\theta,t}}\right\|} = \frac{1}{\left\|\overrightarrow{M_{\theta,t}C_{\theta,t}}\right\|}\cdot\left(0,0,\frac{s\sqrt{3}}{2},0\right)\begin{pmatrix}\cos\theta & 0 & \sin\theta & 0\\ 0 & 1 & 0 & 0\\ -\sin\theta & 0 & \cos\theta & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\cos t & \sin t & 0 & 0\\ -\sin t & \cos t & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}$$

$$= (-\sin\theta\cos t, -\sin\theta\sin t, \cos\theta, 0) \in S_0.$$

Then $\vec{u}$ is the direction of $\overrightarrow{A_{\theta,t}B_{\theta,t}}$, and $\vec{g}$ is the direction of $\overrightarrow{M_{\theta,t}C_{\theta,t}}$.

By (1) and (3), the direction of $\overrightarrow{D_{\theta,t}E_{\theta,t}}$ may be obtained by finding

$$\vec{v} = \frac{\left(\vec{u}\times\vec{g}\times\overrightarrow{n_\theta}\right)}{\left\|\vec{u}\times\vec{g}\times\overrightarrow{n_\theta}\right\|} = \frac{(\cos\theta\cos t, \cos\theta\sin t, \sin\theta, \cos\theta\cos t\tan\theta)}{\sqrt{\cos^2\theta\cos^2 t\tan^2\theta+1}} \in S_\theta.$$

Finally, the direction of $\overrightarrow{P_{\theta,t}Q_{\theta,t}}$ is found by

$$\vec{f} = -\frac{\left(\vec{u}\times\vec{g}\times\vec{v}\right)}{\left\|\vec{u}\times\vec{g}\times\vec{v}\right\|} = \frac{\left(-\cos^2\theta\cos^2 t\tan\theta, -\cos^2\theta\sin t\cos t\tan\theta, -\sin\theta\cos\theta\cos t\tan\theta, 1\right)}{\sqrt{\cos^2\theta\cos^2 t\tan^2\theta+1}}.$$

By (1) and (4), $h\vec{f} = \overrightarrow{P_{\theta,t}Q_{\theta,t}} = Q_{\theta,t} - P_{\theta,t}$, which can be written as

$$h\begin{bmatrix} f_x \\ f_y \\ f_z \\ f_w \end{bmatrix} = \begin{bmatrix} q_x \\ q_y \\ q_z \\ q_w \end{bmatrix} - \begin{bmatrix} p_x \\ p_y \\ p_z \\ p_w \end{bmatrix}.$$

Since $P_{\theta,t} \in S_0$, and $Q_{\theta,t} \in S_\theta$ (i.e. $p_w = 0$ and $q_w = q_x \tan\theta$,) $q_w = hf_w$, $q_x = hf_w / \tan\theta$, and $p_x = hf_w / \tan\theta - hf_x$.

Requiring that $P_{\theta,t}$ remain on the positive x-axis (condition (ii) from part **III**) yields

$$p_y = p_z = 0,\ q_y = hf_y, \text{ and } q_z = hf_z.$$

Altogether, with f as above,

$$Q_{\theta,t} = h\begin{bmatrix} \dfrac{f_w}{\tan\theta} \\ f_y \\ f_z \\ f_w \end{bmatrix} \quad \textit{and} \quad P_{\theta,t} = h\begin{bmatrix} \dfrac{f_w}{\tan\theta} - f_x \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Calculating $M_{\theta,t} = P_{\theta,t} - (h\sqrt{5}/5)\vec{g}$, the vertices of the simplex may then be found as follows:

$$A_{\theta,t} = M_{\theta,t} - \frac{h\sqrt{15}}{5}\vec{u},\quad B_{\theta,t} = M_{\theta,t} + \frac{h\sqrt{15}}{5}\vec{u},\quad C_{\theta,t} = P_{\theta,t} + \frac{2h\sqrt{5}}{5}\vec{g},$$
$$D_{\theta,t} = Q_{\theta,t} - \frac{h\sqrt{15}}{5}\vec{v},\quad E_{\theta,t} = Q_{\theta,t} + \frac{h\sqrt{15}}{5}\vec{v}$$

In particular, notice that A, B, C | | S_0, and D, E | | S_θ, independent of the values of t and θ. Since rotation may take place about the x-w plane and the y-w plane with the simplex staying within its delimiting spaces, unlike the three-dimensional kaleidocycle which only has one degree of rotational freedom, the four-dimensional hyperkaleidocycle has two degrees of rotational freedom.

VI. The Case of Six Regular Simplices

I have thus far exclusively considered hyperkaleidocycles where n was even, and $n \geq 8$. For $n \leq 6$, no twistable hyperkaleidocycle exists.

Consider a hyperkaleidocycle of n regular simplices as described above where n is even. The x-coordinate of P is given by

$$P_x = h(\cos^2\theta\cos^2 t\tan^2\theta + 1)^{-\frac{1}{2}}\left(\frac{1}{\tan\theta} + \cos^2\theta\cos^2 t\tan\theta\right)$$

P is closest to the origin when t = π/2, or $\square$ = π/2 (or both.) For those rotational values, OP = h/tan$\square$. As P is the centroid of equilateral $\square$ ABC, $OP \geq s\sqrt{3}\ /\ 3 = 2h\sqrt{5}\ /\ 5$, or the interiors of several simplices would intersect at the origin. Therefore, OP = h/tan $\square$ $\geq s\sqrt{3}\ /\ 3 = 2h\sqrt{5}\ /\ 5$, or tan$\square$ $\leq \sqrt{5}\ /\ 2$. This inequality is satisfied when n ≥ 8.

The coordinates of Q are given by

$$\square \frac{h}{\square k \tan \square}, \frac{\square h \cos^2 \square \sin t \cos t \tan \square}{k}, \frac{\square h \sin \square \cos \square \cos t \tan \square}{k}, \frac{h}{k} \square$$

where k = $\sqrt{\cos^2 \square \cos^2 t \tan^2 \square + 1}$.

The length OQ is at a minimum when t = $\square$ = 0. Since Q is the midpoint of [DE], it must hold that $OQ \geq s\ /\ 2 = h\sqrt{15}\ /\ 5$ in order for the simplices not to overlap at the origin. For those values of t and $\square$, OQ = h/tan$\square$ = s$\sqrt{15}$ /(6tan$\square$) ≥ s / 2 occurs when tan $\square$ = tan (2π/n) ≤ $\sqrt{15}$ /3. For even n this inequality is satisfied when n ≥ 8. It follows that hyperkaleidocycles consisting of regular simplices must have at least 8 components in order to be rotatable in the manner described above.

VII. The Generalized Hyperkaleidocycle

Based on the previous sections that dealt with regular hyperkaleidocycles I show in this section how a whole class of hyperkaleidocycles can be defined by introducing certain parameters. I will employ the same notation as before. In particular, let n be even, n ≥ 6, and $\square$ = 2π/n.

The positions of the vertices A, B, C, D, and E (I now omit the indices t and $\square$) of a regular simplex in a regular hyperkaleidocycle are determined by the positions of the points P and Q as well as the vectors $\vec{u}$, $\vec{g}$, and $\vec{v}$ (which in turn represent the directions of the vectors $\overrightarrow{AB}$, $\overrightarrow{CM}$, and $\overrightarrow{DE}$):

$$A = P \square \frac{h\sqrt{5}}{5}\vec{g} \square \frac{h\sqrt{15}}{5}\vec{u}, \qquad B = P \square \frac{h\sqrt{5}}{5}\vec{g} + \frac{h\sqrt{15}}{5}\vec{u}, \qquad C = P + \frac{2h\sqrt{5}}{5}\vec{g},$$

$$D = Q \square \frac{h\sqrt{15}}{5}\vec{v}, \qquad E = Q + \frac{h\sqrt{15}}{5}\vec{v}.$$

The normed vectors $\vec{u}$, $\vec{g}$, and $\vec{v}$ were scaled so that ABCDE was a regular simplex. If instead, simplex ABCDE is defined so that

$$A = P \square \square\vec{g} \square \square\vec{u}, \qquad B = P \square \square\vec{g} + \square\vec{u}, \qquad C = P + \square\vec{g}$$

$$D = Q \square \square\vec{v}, \qquad E = Q + \square\vec{v}$$

with arbitrary $(\square,\square,\square,\square,\square,\square,\square) \square^{7}$, then ABCDE is still a (not necessarily regular) simplex with A, B, C | | S_0, and D, E | | $S_{\square}$. Furthermore, (1) from part **I** still holds.

By placing further simplices that are equivalent to ABCDE in the same manner as in **IV**, a closed ring is again obtained where neighboring simplices alternate between sharing one common edge, or one common face.

In order for such a hyperkaleidocycle to be rotatable, the following inequalities must hold:

$$|\square|,|\square|\ \square\ h/\tan\square,\ |\square|\ \square\ h/\tan\square,\ \text{and}\ \sqrt{\square^2+\square^2},\sqrt{\square^2+\square^2}\ \square\ h/\tan\square.$$

If not, there are positions of the hyperkaleidocycle for which several tetrahedra intersect at the origin (see **VI**.)

A hyperkaleidocycle with n components built by symmetry from one simplex ABCDE with

$$A = P\ \square\ \square\vec{g}\ \square\ \square\vec{u}, \qquad B = P\ \square\ \square\vec{g} + \square\vec{u}, \qquad C = P + \square\vec{g}$$

$$D = Q\ \square\ \square\vec{v}, \qquad E = Q + \square\vec{v}$$

where

$$|\square|,|\square|,|\square|,\sqrt{\square^2+\square^2},\sqrt{\square^2+\square^2}\ \square\ h/\tan\square$$

is called a *normal hyperkaleidocycle*. By the definition of P, Q, $\vec{u}$, $\vec{g}$, and $\vec{v}$, simplices that are components of normal hyperkaleidocycles have the following (in the context of this paper crucial) property: the two edges [AB] and [DE], the altitude to C of □ ABC, and the segment joining the midpoint of [DE] to the centroid of □ ABC are all pairwise orthogonal.

VIII. The Closed Hyperkaleidocycle

Using the parameters $\square,\square,\square,\square,\square,\square,\square$ a variety of different forms and types of hyperkaleidocycles can be designed. In particular, when

$$n = 6,\ \text{so}\ \square = \frac{\square}{3},\ \square = \square = \frac{h}{\tan\square},\ \square = \square = \frac{h}{2\tan\square},\ \square = \square = \frac{h\sqrt{3}}{2\tan\square},\ \text{and}\ \square = \frac{h}{\tan\square}$$

the result is a *closed hyperkaleidocycle* with the property that for various combinations of rotations in $\square$ and t, vertices of several simplices meet at the origin, and thus the central hole in the ring closes in these positions.

In particular, the following vertices are at the origin for the listed rotations of $(\square,t)$:

$$A{:}\,\square\frac{\square}{\square 2},\frac{4\square}{3}\square\ \square\frac{\square 3\square}{\square 2},\frac{5\square}{3}\square\quad B{:}\,\square\frac{\square}{\square 2},\frac{2\square}{3}\square\ \square\frac{\square 3\square}{\square 2},\frac{\square}{3}\square\quad C{:}\,\square\frac{\square}{\square 2},0\square\ \square\frac{\square 3\square}{\square 2},\square\square\quad D{:}(0,0),(\square,\square)\quad E{:}(0,\square),(\square,0).$$

IX. Face-Only Hinging

It is possible to construct a hyperkaleidocycle in which all hinges are pairs of adjacent faces (rather than those described above, where hinges alternate between pairs of adjacent faces and pairs of adjacent edges.)

Using the notation from above, if vertex C is required to remain in the y-z plane, it is both a member of spaces S_0 and $S_\square$. With this restriction, face □ ABC remains in S_0, and face □ CDE remains in $S_\square$ as the hyperkaleidocycle rotates. In particular, the reflection of simplex □ about the space $S_\square$ (as described in **IV**) leaves vertex C unmoved. Thus the faces □ ABC and □ CDE become the hinges between □ and its adjacent simplices.

From **VII**, $C = P + \square\vec{g}$. Using the derivations in **V**, this may be written as:

$$\begin{bmatrix} C_x \\ C_y \\ C_z \\ C_w \end{bmatrix} = \begin{bmatrix} P_x \\ 0 \\ 0 \\ 0 \end{bmatrix} + \square \begin{bmatrix} g_x \\ g_y \\ g_z \\ g_w \end{bmatrix} = h \begin{bmatrix} \frac{f_w}{\tan\square} \square f_x \\ 0 \\ 0 \\ 0 \end{bmatrix} + \square \begin{bmatrix} \square\sin\square\cos t \\ \square\sin\square\sin t \\ \cos\square \\ 0 \end{bmatrix}.$$

C remains in the y-z plane, therefore when $C_x = 0$, or

$$h\left(\frac{f_w}{\tan\square} \square f_x\right) \square \square\sin\square\cos t = 0 \quad \square \quad h\left(\frac{1}{\tan\square} + \cos^2\square\cos^2 t\tan\square\right) \square \square\sin\square\cos t = 0,$$

which is a quadratic in $\cos t$ with A = $h\tan\square\cos^2\square$, B = $\square\square\sin\square$, and $C = h / \tan\square$.

Consider the solutions

$$\cos t = \frac{\square\sin\square \pm \sqrt{\square^2\sin^2\square \square 4h\cos^2\square}}{2h\tan\square\cos^2\square}. \qquad (5)$$

There is no value of $\square$ for which (5) is continuous (and real) as $\square^2\sin^2\square \square 4h\cos^2\square \geq 0$ only when $\square^2 / (\square^2 + 4h) \geq \cos^2\square$. The result is that while a hyperkaleidocycle hinged only at faces is possible, and while such a ring has, like its three-dimensional analog, only one degree of rotational freedom, it cannot rotate continuously, but will always eventually get “stuck”.

X. Further Research

A variety of questions remain to be researched. Can a ring of other regular or irregular polytopes be formed where all hinges are face-to-face and that doesn’t get “stuck” as it rotates? What would be the four-dimensional analog to the ring of Yoshi cubes, and would it have similar properties to its three-dimensional counterpart? Is the procedure for producing a regular hyperkaleidocycle out of simplices generalizable to higher dimensions?

I leave these exercises to be explored by the reader.